Generalized persistence exponents: an exactly soluble model

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It was recently realized that the persistence exponent appearing in the dynamics of nonequilibrium systems is a special member of a continuously varying family of exponents, describing generalized persistence properties. We propose and solve a simplified model of coarsening, where time intervals between spin flips are independent, and distributed according to a Lévy law. Both the limit distribution of the mean magnetization and the generalized persistence exponents are obtained exactly.

The surprise caused by the discovery of new nontrivial exponents in the dynamics of simple nonequilibrium systems [1,2] motivated a long series of works, mainly devoted to the search of simple models or experimental situations, where the so called persistence exponents could be computed or measured [3–10]. More recently, the idea of *persistent large deviations* [11] led to the introduction of families of new nontrivial persistence exponents in, e.g., the one dimensional Glauber-Ising chain at zero temperature, or the simple diffusion equation. The probability of persistent deviations, defined as the probability that the mean magnetization $M_t = t^{-1}\int_0^t \mathrm{d}u\,\sigma(u)$ of the spin (or of the sign of the diffusing field) at a given site was, for all previous times, greater than some level x, where $-1 \le x \le 1$, was observed to decay algebraically at large times, with an exponent $\theta(x)$ continuously varying with x. When $x = 1$, this probability is the usual persistence probability, since imposing $M_t = 1$ requires that the spin never flipped. Hence $\theta(1) = \theta$, the usual persistence exponent. Furthermore, the distribution of M_t does not peak around zero for $t \to \infty$, but tends to a nontrivial limit distribution on $[-1, 1]$, singular at both ends as $(1 \mp x)^{\theta-1}$. For coarsening systems, computing the exact value of θ turns out to be a hard problem, so one does not expect the computation of $\theta(x)$, or even of the distribution of the mean magnetization, to be easily reachable. The origin of the difficulty is that spins at different sites are strongly correlated.

The aim of this letter is to present an extremely simplified version of the coarsening models mentioned above, which allows for exact analytical expressions both of the limit distribution of the mean magnetization M, and of the generalized persistence exponents $\theta(x)$. Despite its simplicity, this model retains the essential features of the coarsening process, in particular its non stationary properties, as will be discussed below.

In this model, which describes the dynamics of a single spin, we assume that the time intervals between spin flips are independent. It is indeed intuitively clear that, because of the ever growing size of domains in coarsening systems (or of the diffusion length in the diffusion equation), a spin at a given site can remain in the same direction for a very long time before a domain wall crosses this particular point and flips the spin in the reversed direction. By definition of the persistence exponent θ, the time τ before a spin is flipped is very broadly distributed, with a power law tail decaying as $\tau^{-1-\theta}$ for large τ. The simplest approximation is therefore to neglect the correlations between the different time intervals between flips, all assumed to be distributed with the same density $p(\tau)$, decaying as $\tau^{-1-\theta}$.

For simplicity, the distribution of time intervals $p(\tau)$ is chosen to be a positive stable Lévy distribution of index $0 < \theta < 1$ denoted by $L^b_\theta(\tau)$. (The case $\theta > 1$ will be discussed below.) Its Laplace transform reads $\hat{L}^b_\theta(s) = \exp(-bs^\theta)$, where b is the scale factor of the distribution, i.e. the typical values of τ are of order $b^{1/\theta}$ [12]. As is well known, $L^b_\theta(\tau)$ decays asymptotically as $\tau^{-1-\theta}$ [12]. We will always suppose that $\sigma(t = 0) = 1$. On the time axis, the process thus defined is a renewal process.

We have investigated the statistics of the process, both *after n sign changes*, or *at time t*, with very similar results in the asymptotic regime. After n sign changes, the time elapsed and the magnetization of the spin read

$$t_n = \tau_1 + \tau_2 + \cdots + \tau_n, \tag{1}$$
$$S_n \equiv t_n M_n = \tau_1 - \tau_2 + \cdots + (-)^{n-1}\tau_n, \tag{2}$$

while, at time t, they are given by

$$t = t_{N_t} + \lambda, \quad S_t \equiv tM_t = S_{N_t} + (-)^{N_t}\lambda. \tag{3}$$

In the first case, n is given and t_n is a random variable, while in the second one, t is given and N_t is the random variable equal to the largest n for which $t_n \le t$. Finally λ is the length of time measured backwards from t to the last event. The corresponding distributions are defined as

$$P(n, x) = P\left(M_n = S_n/t_n \geq x\right), \tag{4}$$

$$P(t, x) = P\left(M_t = S_t/t \geq x\right). \tag{5}$$

For distributions which are peaked around their means at large times, these quantities are referred to as the probabilities of large deviations and are exponentially decreasing with n or t respectively. In the present case, where $p(\tau) = L_\theta^b(\tau)$ is a positive Lévy distribution, we find the limit distribution

$$P(x) = \lim_{n\to\infty} P(n, x) = \lim_{t\to\infty} P(t, x), \tag{6}$$

$$= \frac{1}{\pi\theta}\left[\frac{\pi}{2} - \arctan\left(\frac{r\omega^{-\theta} + \cos\pi\theta}{\sin\pi\theta}\right)\right]. \tag{7}$$

where $\omega = (1-x)/(1+x)$ and $r = 1$ (see below).

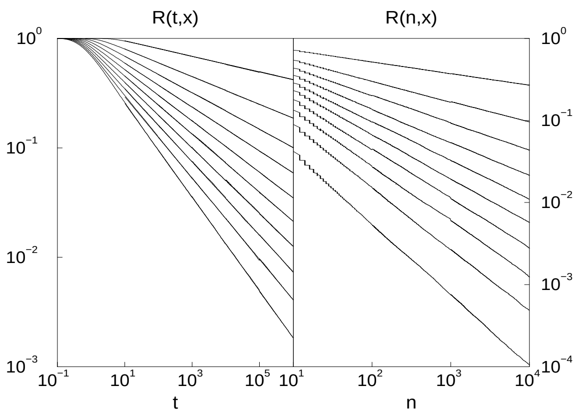

FIG. 1. Plot of $R(t, x)$ (left) and $R(n, x)$ (right) for $\theta = 1/2$ and various values of x, in log-log coordinates. The power-law behavior of both quantities for large times is clearly seen.

Let us sketch the proof of (7) for $P(n, x)$, leaving the calculation of $P(t, x)$ to a longer publication [13]. We introduce T_n^+ and T_n^-, which are the lengths of time spent by the spin, respectively in the positive or negative direction, such that $t_n = T_n^+ + T_n^-$ and $S_n = T_n^+ - T_n^-$, with $T_n^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k+1}$, if $n = 2k+1$, and $T_n^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k-1}$, if $n = 2k$, and $T_n^- = \tau_2 + \tau_4 + \cdots + \tau_{2k}$, in both cases. Then $P(S_n/t_n \geq x) = P(T_n^-/T_n^+ \leq \omega)$ with $\omega = (1-x)/(1+x)$. Since T_n^+ and T_n^- are sums of stable Lévy random variables L_θ^b, they are themselves stable Lévy random variables $L_\theta^{b^\pm}$, where, using the addition rule of the scale parameters, $b^- = kb$, and $b^+ = kb$ (if $n = 2k$), or $b^+ = (k+1)b$ (if $n = 2k+1$). The determination of $P(n, x)$ therefore amounts to computing the distribution of the ratio of two Lévy laws with parameters b^- and b^+. Denoting by H the Heaviside function, and using its Laplace representation along the Bromwich contour, one finds

$$P(T_n^-/T_n^+ > \omega) = \int_0^\infty \mathrm{d}\tau_1 \mathrm{d}\tau_2 L_\theta^{b^+}(\tau_1) L_\theta^{b^-}(\tau_2) H\left(\frac{\tau_2}{\tau_1} - \omega\right)$$

$$= \int \frac{\mathrm{d}s}{2i\pi s} \exp[-b^+(s\omega)^\theta] \exp[-b^-(-s)^\theta].$$

This integral leads to (7) with $r = b^-/b^+$. In the limit $n \to \infty$, $r \to 1$. This derivation also shows that whenever n is even, $P(n, x) = P(x)$.

The limit density $f(x) = -P'(x)$ of the mean magnetization reads

$$f(x) = \frac{\sin\pi\theta}{2\pi} \frac{2 + \omega + \omega^{-1}}{2\cos\pi\theta + \omega^\theta + \omega^{-\theta}}. \tag{8}$$

It is even, and diverges when $x \to \pm 1$ as $(1 \mp x)^{\theta-1}$. For $\theta < \theta_c = 0.5946\ldots$, where θ_c is the solution of $\theta_c = \cos(\pi\theta_c/2)$, $x = 0$ corresponds to a minimum of $f(x)$, while for larger θ, it corresponds to a local maximum. This can be interpreted as a precursory sign of the fact that $f(x)$ tends to $\delta(x)$ for $\theta > 1$. (It also shows that $f(x)$ cannot be approximated by a beta distribution when θ is too large. In this respect, compare to the discussion in [14].)

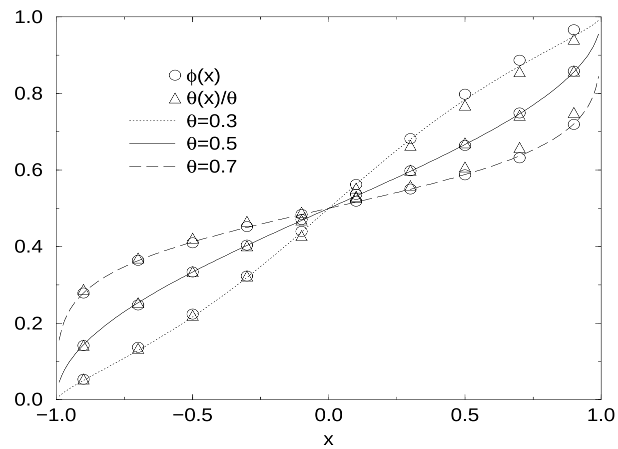

FIG. 2. Plot of the exponents $\phi(x)$ and $\theta(x)/\theta$ for $\theta = 0.3, 0.5, 0.7$, showing that the relation $\theta(x) = \theta\phi(x)$ holds. The lines corresponds to the exact result $\phi(x) = 1 - P(x)$.

We now consider the probability of persistent large deviations, defined as the probability that the mean magnetization M was, for all previous times, greater than some level x. More precisely one defines the quantities $R(n, x) = P(M_{n'} \geq x, \forall n' \leq n)$ and similarly $R(t, x) = P(M_{t'} \geq x, \forall t' \leq t)$. Numerical computations show that both quantities decay algebraically in the asymptotic regime (see Fig. 1), respectively as

$$R(n, x) \sim n^{-\phi(x)}\ (n \gg 1), \quad R(t, x) \sim t^{-\theta(x)}\ (t \gg 1),$$

where the two families of exponents are related by $\theta(x) = \theta\phi(x)$ (see Fig. 2). This relation is indeed expected since for a given n, t_n scales as $n^{1/\theta}$. Note that by definition of the model, $\theta(1) = \theta$. We also observe with very good accuracy (see Fig. 2) the relation

$$\phi(x) = 1 - P(x) = \int_{-1}^x \mathrm{d}u f(u), \tag{9}$$

which we now establish exactly.

For this, we note that $R(n,x)$ is the joint probability that $S_{n'} \geq x t_{n'}$ for all $1 \leq n' \leq n$. Since clearly $R(2k,x) = R(2k+1,x)$, we assume that n is even, and write

$$R(n=2k,x) = P(\xi_1 \geq 0, \xi_1+\xi_2 \geq 0, \dots, \xi_1+\xi_2+\cdots+\xi_k \geq 0), \tag{10}$$

where $\xi_i = (1-x)\tau_{2i-1} - (1+x)\tau_{2i}$. Since the τ_i are positive Lévy variables of index θ, the ξ_i are also Lévy variables of index θ, with an asymmetry parameter $\beta = (\omega^\theta - 1)/(\omega^\theta + 1)$, which measures the relative weight of the negative and positive tails [12]. The solution to (10) for general stable Lévy variables is known [20,15]. It reads

$$R(n=2k,x) = \frac{\Gamma(k+1-q)}{\Gamma(k+1)\Gamma(1-q)}, \tag{11}$$

where $1-q$ is the probability that ξ is positive. This probability is precisely the quantity $P(n=2,x)$ introduced above, itself equal to $P(x)$. Hence $q = 1-P(x)$. Finally, the large k behavior of the r.h.s. of (11) is $\propto k^{-q}$, i.e., $\phi(x) = q$, which completes the proof of (9). Note that the plot of (11) is indistinguishable from that obtained numerically for $R(n,x)$. Eqs. (7), (9), and (11) are the main results of this work.

The rest of this paper is devoted to a discussion of further properties of the model, and to a presentation of some possible generalizations.

First, the stochastic process presented above, where time intervals between spin flips are independent and distributed according to a Lévy distribution, exhibits nontrivial temporal properties, both from mathematical [15,12], and physical [16–18] points of view. For example, although $p(\tau)$ is fixed in time, the probability distribution of the length of time $\tilde{\lambda}$ from some time origin (or waiting time) t_{w} to the next flip is non stationary for $\theta < 1$, i.e. it depends both on t_{w} and $\tilde{\lambda}$, while it is asymptotically independent of t_{w} for $\theta > 1$. As a consequence, the probability that a given spin did not flip between times t_{w} and $t_{\mathrm{w}} + t$ is a function of t_{w}/t if $\theta < 1$, while it is independent of t_{w} if $\theta > 1$ [16]. Thus for $\theta < 1$, this model captures the *aging* [19] nature of the persistence phenomenon. This property is deeply related to the fact that the largest τ_i in the sum $t_n = \sum_{i=1}^n \tau_i$ contributes to a finite fraction of t_n for $\theta < 1$ even in the limit $n \to \infty$, while this fraction is asymptotically zero for $\theta \geq 1$ [17,18]. Correspondingly, this also ensures that the distribution of the mean magnetization does not peak around $x = 0$, as was shown above.

Despite its simplicity, the model discussed here thus shares many features of more complex coarsening processes. As shown above, it leads to nontrivial predictions for the quantities $P(x)$ and $\theta(x)$. Also, the behavior of $R(t,x)$ observed in Fig. 1 strongly resembles that found in [11] for the Glauber-Ising chain or the diffusion equation. These predictions can be seen as approximations for these more general models. In Fig. 3, we compare, for the Glauber model at zero temperature, the function $\theta_{\mathrm{G}}(x)/\theta_{\mathrm{G}}(1)$, as determined numerically in [11], both with $1-P(x)$, where $P(x)$ is given by (7) with $\theta = 3/8$ [3], and with $1-P_{\mathrm{G}}(x)$, the distribution of magnetization measured numerically in [11]. Although there is qualitative agreement between the three curves, the above relations are clearly only approximate. Furthermore there remains to understand the qualitative difference in behavior between the persistence exponents $\theta(x)$ for diffusion and for the Glauber-Ising chain. It would be therefore interesting to generalize the present model to include some correlations between the time intervals τ_i. The independent interval model presented here is actually, in many respects, similar to the random energy model (REM) for spin-glasses [17,18]. For example, the r.h.s. of (11) is identical to the expression for the participation ratio Y_{k+1} in the REM, with a reduced temperature equal to q [17,18,13].

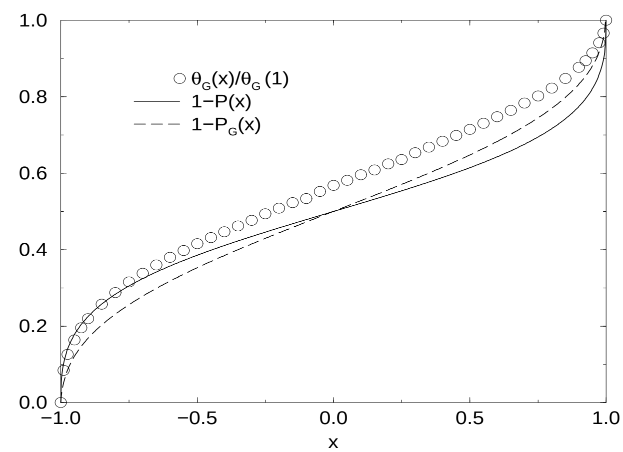

FIG. 3. Comparison between the function $\theta_{\mathrm{G}}(x)/\theta_{\mathrm{G}}(1)$ for the Glauber-Ising model in one dimension, as determined numerically in [11], and two predictions inspired from the present model: $1-P(x)$, where $P(x)$ is given by (7) with $\theta = 3/8$, and $1-P_{\mathrm{G}}(x)$ (see [11]).

The only feature of $p(\tau)$ relevant for the results given here is its asymptotic power-law behavior at large values of n or t, and not its detailed behavior for small τ. The reason is that the sum of a large number of power-law distributed variables converges (for $\theta < 1$) towards the Lévy distributions considered here. In our simulations we used two different distributions both decaying as $\tau^{-1-\theta}$ for large τ, obtaining the same results for sufficiently large times. Note that the relations $\phi(x) = \theta(x)/\theta = 1-P(x)$ are best obeyed numerically for $\theta = 1/2$, because in this case it is easy to generate the corresponding stable distribution.

We also studied the case $\theta > 1$, where $p(\tau)$ has a

finite first moment. In this case, it is easy to check that t_n grows linearly with n, while S_n grows as $n^{1/\theta}$ for $1 < \theta < 2$ and as $\sqrt{n}$ for $\theta > 2$ [12]. Hence, the quantity M_n tends to zero for large n, and $f(x)$ collapses to a δ function. However, the persistence exponents $\theta(x)$ remain well defined, and are found to be equal to $\theta(x > 0) = \theta$, $\theta(x = 0) = 1/2$ and $\theta(x < 0) = 0$. This shows that the relation between $\theta(x)$ and $P(x)$ actually still holds in this degenerate case, except for $x = 0$ where the value of $P(x)$ is ill defined. However, the nature of the persistence phenomenon in this model is quite different when $\theta > 1$, where it becomes stationary (see above). It would be interesting to see if this is also true of more general models where $\theta > 1$, such as the diffusion equation in high dimensions [4,11,14].

We have generalized slightly the problem, by choosing the time intervals during which the spin σ is respectively equal to 1 or to -1 with a different scale factor. The distribution $f(x)$ becomes asymmetrical. One can however check, both numerically and analytically, that the relations $\theta(x)/\theta = \phi(x) = q = 1 - P(x)$ still hold in this case. We have also relaxed the deterministic alternation of signs, and considered $S_n = \sum_{i=1}^{n} a_i \tau_i$, where the a_i are independent identically distributed random variables, with $\langle a \rangle = 0$ and $\langle a^2 \rangle$ finite. The above results (7) and (9) for the limit distribution and the exponents remain unchanged.

When $\theta = 1/2$, $p(\tau) = L^b_{1/2}(\tau)$ is precisely the distribution of the time intervals between two returns to the origin of the binomial random walk with equal steps ± 1, in the regime of long times. In this sense the binomial random walk is 'primitive' with respect to the present 'walk' with time-space coordinates t, S_t (or t_n, S_n), and instantaneous velocity equal to $\sigma(t)$. For the latter, $y = T^+_t/t$, is the fraction of time spent by the walk stepping in the positive direction, or for the primitive walk, the fraction of time spent on the positive half axis. Its distribution is well known, and given at large times by the arc sine density $1/\pi\sqrt{y(1-y)}$, which is precisely the result (8), with $\theta = 1/2$, and $x = S_t/t = 2T^+_t/t - 1$. In this respect, (8) appears as a generalization of the arc sine law. A striking consequence of the present work is the existence of the families of exponents $\theta(x)$ and $\phi(x)$, since, when $\theta = 1/2$, these exponents describe properties of the simple random walk. We note that the present work provides a clue to the determination of the temporal behavior of the hierarchy of quantities introduced in [11].

Finally, it is tempting to conjecture that for a generic stochastic process S_t such that there exist an $\alpha > 0$ for which S_t/t^α admits a nontrivial limit distribution at large times, then

$$P\left(\frac{S_{t'}}{t'^\alpha} \geq x, \forall t' < t\right) \propto t^{-\theta(x)} \qquad (12)$$

Such a conjecture, verified in our model, is also corroborated by the works of [21,22] on the random walk (for which $\alpha = 1/2$). In this case x is not restricted to a finite interval, and $\theta(x)$ cannot be simply expressed in terms of the Gaussian distribution of $S_t/\sqrt{t}$.

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